In-Class Exam #3 Review Sheet, Covers 11.5-11.10
Math 280, Vanden Eynden

In #1 – #9, determine whether the series converges conditionally, converges absolutely, or diverges. Name any relevant theorems, tests, facts and/or mathematical reasoning you used to reach your conclusion. If applicable, make sure to **show** that the series meets the conditions of the test you use.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n+1}}{4^{3n}}$ 2. $\sum_{n=1}^{\infty} \frac{5^n}{n^2 4^{n+1}}$ 3. $\sum_{n=1}^{\infty} \frac{2n^2+1}{n!}$

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{2n}-3}$ 5. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{4+4^n}$ 6. $\sum_{n=1}^{\infty} \frac{2\cdot 5\cdot 8\cdots(3n-1)}{3\cdot 5\cdot 7\cdots(2n+1)}$

7. $\sum_{n=1}^{\infty} \frac{\sin\sqrt{n}}{n^2}$ 8. $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$ 9. $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

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10. a. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}$ is convergent.

b. How many terms of the series do you need to add in order to find the sum with an error less than 0.001?

c. Approximate the sum of this series accurate to 3 decimal places.

In #11 – #13, find the radius of convergence and the interval of convergence for the power series.

11. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2 5^n}$ 12. $\sum_{n=1}^{\infty} \frac{2^n (x-2)^n}{(n+2)!}$ 13. $\sum_{n=1}^{\infty} n^n (x+1)^n$

In #14 – 16, find a power series representation for the function $f(x)$ and state its radius of convergence.

14. $f(x) = \frac{5}{1-4x^2}$ 15. $f(x) = \frac{x^3}{(2+x)^2}$ 16. $f(x) = \ln(x+5)$

In problems#17–21, find the Maclaurin series for f and its radius of convergence. To find each power series, you may use either the direct method (definition of a Maclaurin series, taking several derivatives and finding a pattern) or use known series such as geometric series, binomial series or the Maclaurin series shown in Section 11.10, Table 1, pg 762.

17. $f(x) = 2xe^{3x}$ 18. $f(x) = \sin(x^4)$ 19. $f(x) = x\cos(2x^2)$

20. $f(x) = 3^x$ 21. $f(x) = \sqrt{1+x^4}$

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22. Use the Maclaurin series found in #21 to approximate $\int_0^1 \sqrt{1+x^4}\, dx$ correct to 2 decimal places.

23. Find the taylor series for $f(x) = \frac{1}{x}$ centered at $a = -3$. Also f ind its radius of convergence, R.

(A) Geometric Series: The series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$ is convergent if $|r| < 1$ and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If $|r| \geq 1$, the series is divergent.

(B) Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(C) Integral Test: Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words,

(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(D) P-series: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

(E) Remainder Estimate for the Integral Test: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = S - S_n$, then $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)dx$.

(F) Series Sum Estimate for the Integral Test: $s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_n^{\infty} f(x)dx$

The midpoint of this interval is an estimate of *s*, with error < (half the interval's length).

(G) The Comparison Test: If $\sum a_n$ and $\sum b_n$ are series with positive terms and

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

(H) The Limit Comparison test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite number and $c > 0$, then either both series converge or both diverge.

(I) The Alternating Series Test:

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + ...$ where $b_n > 0$ satisfies

(i) $b_{n+1} \le b_n$

(ii) $\lim_{n\to\infty} b_n = 0$

then the series converges.

(J) Alternating Series Estimation Theorem:

If $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

(i) $0 \le b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

Then $|R_n| = |s - s_n| \le b_{n+1}$

(K) Absolute Convergence:

If $\sum |a_n|$ converges, then $\sum a_n$ converges (absolutely).

(L) The Ratio Test for Absolute Convergence:

Let $\sum a_n$ be a series with non-zero terms and suppose $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$;

i. If L < 1, the series $\sum a_n$ is absolutely convergent.

ii. If L > 1 or $L = \infty$, then the series $\sum a_n$ diverges.

iii. If L = 1 , the test is inconclusive. The series may be convergent or divergent. Use another test (not The Root Test)

(M) The Root Test for Absolute Convergence:

Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

i. If L < 1, then the series $\sum a_n$ is absolutely convergent.

ii. If L > 1 or $L = \infty$, then the series $\sum a_n$ diverges.

iii. If L = 1 , the test is inconclusive. The series may be convergent or divergent. Use another test (not The Ratio Test)